Important Theorems and Definitions concerning Limits and Continuity

The Definition of limit (delta-epsilon definition)	The limit of $f(x)$ as x approaches c is L, written $\lim_{x\to c} f(x) = L$ if and only if f is defined for all x on some open interval containing c (except possibly at c itself) and for any $\varepsilon > 0$ (no matter how small) there exists a $\delta > 0$ such that $\lvert f(x) - L \rvert < \varepsilon$ whenever $0 < \lvert x - c \rvert < \delta$
Properties of Limits	1. Constant Law: $\lim_{x\to c} b = b$ 2. Identity Law: $\lim_{x\to c} x = c$ 3. Scalar Law: $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ 4. Sum or difference law $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$ 5. Product Law $\lim_{x\to c}\left(f(x)g(x)\right) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$ 6. Quotient Law $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided $\lim_{x\to c} g(x) \neq 0$ 7. Power Law $\lim_{x\to c}\left(f(x)\right)^n = \left(\lim_{x\to c} f(x)\right)^n$
Limit of a Composite Function	If $f(x)$ and $g(x)$ are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$,. Then $\lim_{x\to c} f\left(g(x)\right) = f\left(\lim_{x\to c} g(x)\right) = f(L)$
Limits of Trigonometric Functions	If c is a real number in the domain of a trig function, then 1. $\lim_{x\to c}\sin(x) = \sin(c)$ 2. $\lim_{x\to c}\cos(x) = \cos(c)$ 3. $\lim_{x\to c}\tan(x) = \tan(c)$ 4. $\lim_{x\to c}\csc(x) = \csc(c)$ 5. $\lim_{x\to c}\sec(x) = \sec(c)$ 6. $\lim_{x\to c}\cot(x) = \cot(c)$
The Almost Equal Theorem	If $f(x) = g(x)$ for all $x \neq c$ in some open interval containing c, and $\lim_{x\to c} g(x) = L$ then $\lim_{x\to c} f(x) = L$
The Squeeze (or Sandwich) Theorem	If $f(x) \le g(x) \le h(x)$ for all x in an open interval containing c, except possibly at c itself, and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$

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Two Special Trigonometric Limits AKA The "Necklace, Earring, bracelet" theorems	$\lim_{x\to 0}\frac{\sin x}{x}=1$ and $\lim_{x\to 0}\frac{1-\cos x}{x}=0$
Definition of one sided limits	The limit of $f(x)$ as x approaches c from the right is L, written $\lim_{x\to c+} f(x)=L$ if and only if f is defined for all x on some open (c,b) and for any $\varepsilon>0$ (no matter how small) there exists a $\delta>0$ such that $\lvert f(x)-L\rvert<\varepsilon$ whenever $0<x-c<\delta$ The limit of $f(x)$ as x approaches c from the left is L, written $\lim_{x\to c-} f(x)=L$ if and only if f is defined for all x on some open (b,c) and for any $\varepsilon>0$ (no matter how small) there exists a $\delta>0$ such that $\lvert f(x)-L\rvert<\varepsilon$ whenever $0<c-x<\delta$
Existence of a Limit theorem	The $\lim_{x\to c} f(x)=L$ if and only if $\lim_{x\to c-} f(x)=\lim_{x\to c+} f(x)=L$
Definition of Continuity of a function at a point	A function is continuous at x = c if and only if the following three conditions are true i) $f(c)$ exists ii) $\lim_{x\to c} f(x)$ exists iii) $\lim_{x\to c} f(x)=f(c)$
Definition of continuity on an <u>open</u> and <u>closed</u> Interval	A function is continuous on an open interval (a,b) if it is continuous at each point in the open interval (a,b). A function is continuous on a closed interval [a,b] if the function is continuous on the open interval and i) $f(a)$ exists; ii) $\lim_{x\to a+} f(x)$ exists; iii) $\lim_{x\to a+} f(x)=f(a)$ and i) $f(b)$ exists; ii) $\lim_{x\to b-} f(x)$; iii) $\lim_{x\to b-} f(x)=f(b)$
The Intermediate Value Theorem	If *f* is continuous of the closed interval [*a*,*b*] and *d* is any number between *f*(*a*) and *f*(*b*), then there exists at least one number *c* in (*a*,*b*) such that *f*(*c*)=*d*.
Definition of Infinite Limits	The limit of $f(x)$ as x approaches c is infinity, written $\lim_{x\to c} f(x)=\infty$ if and only if f is defined for all x on some open interval containing c (except possibly at c itself) and for any $M>0$ (no matter how large) there exists a $\delta>0$ such that $f(x)>M$ whenever $0<\lvert x-c\rvert<\delta$

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Definition of Infinite Limits (continued)	The limit of $f(x)$ as x approaches c is negative infinity, written $\lim_{x \to c} f(x) = -\infty$ if and only if f is defined for all x on some open interval containing c (except possibly at c itself) and for any $N < 0$ (no matter how small) there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < \lvert x - c \rvert < \delta$. Note: If the $\lim_{x \to c} f(x) = \infty$ or $\lim_{x \to c} f(x) = -\infty$ then the limit DOES NOT EXIST. The notation means that the limit fails to exist because it increases (or decreases) without bound.
Definition of Vertical Asymptotes	The graph of the function $y = f(x)$ has a vertical asymptote at $x = c$ if and only if f(x) approaches infinity (or negative infinity) as x approaches c from the right or the left.
Properties of Infinite Limits	If $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$ then 1. Sum or difference $\lim_{x \to c} [f(x) \pm g(x)] = \infty$ 2. Product $\lim_{x \to c} \left(f(x) g(x) \right) = \infty$ if $L > 0$ and $\lim_{x \to c} \left(f(x) g(x) \right) = -\infty$ if $L < 0$ 3. Quotient $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$ Similar properties hold for one-sided limits and for functions for which the $\lim_{x \to c} f(x) = -\infty$
Definition of Limits at Infinity	The limit of f(x) as x approaches infinity is L, written $\lim_{x \to \infty} f(x) = L$ if for any $\varepsilon > 0$, there exists an $M > 0$ such that $\lvert f(x) - L \rvert < \varepsilon$ whenever $x > M$. The limit of f(x) as x approaches negative infinity is L, written $\lim_{x \to -\infty} f(x) = L$ if for any $\varepsilon > 0$, there exists an $N < 0$ such that $\lvert f(x) - L \rvert < \varepsilon$ whenever $x < N$.
Definition of a Horizontal Asymptote	The graph of the function $y = f(x)$ has a horizontal asymptote at $x = L$ if and only if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.